Loop Representation of charged particles interacting with Maxwell and Chern-Simons fields

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The loop representation formulation of non-relativistic particles coupled with abelian gauge fields is studied. Both Maxwell and Chern-Simons interactions are separately considered. It is found that the loop-space formulations of these models share significant similarities, although in the Chern-Simons case there exists an unitary transformation that allows to remove the degrees of freedom associated with the paths. The existence of this transformation, which allows to make contact with the anyonic interpretation of the model, is subjected to the fact that the charge of the particles be quantized. On the other hand, in the Maxwell case, we find that charge quantization is necessary in order to the geometric representation be consistent.

I. INTRODUCTION

The loop representation (L.R.) constitutes an useful tool in present day investigations in gauge theories [1,2]. There are several approaches to the L.R. [3–6], all of them sharing the recognition of string-like structures as the basic objects needed to build a geometric representation for gauge field quantization.

In this paper we study the L.R. formulation of point particles interacting with abelian gauge fields. The coupling of point particles to fields presents certain subtleties that make the canonical quantization far from being straightforward. In turn, the corresponding L.R. shows its own particularities, which had not yet been reported. This study is carried out first for non-relativistic dynamical point particles in electromagnetic interation. We shall not worry about the lack of Lorentz covariance, neither we shall discuss regularization issues. As we shall see, for the L.R. formulation of this model to be consistent, charge must be quantized. This result should be compared with a similar one obtained several years ago for the Maxwell theory, within the Spin Networks version of the L.R. [7,8].

As a second model we consider the topological interaction between non-relativistic dynamical charged particles caused by a Chern-Simons term [9,10]. Althought both theories share the same geometrical framework when quantized in the L.R., in the Chern-Simons case the loop dependence may be eliminated by means of an unitary transformation, which yields a quantum mechanics of many particles subjected to a long range interaction. As we shall discuss, this unitary transformation holds provided charge is quantized.

This paper is organized as follows. In section II we study the L.R. formulation of non-relativistic point particles in electromagnetic interaction. Section III is devoted to consider the L.R. quantization of point particles with Chern-Simons interaction. Some final remarks are left for the last section.

II. ELECTROMAGNETIC INTERACTION OF NON-RELATIVISTIC POINT PARTICLES

The action for N electromagnetically interacting non-relativistic charged particles may be writen as

$$S = \int dt \sum_{p=1}^{N} \Big(\frac{1}{2}\, m_{(p)} |\dot{\vec{r}}_{(p)}|^2 - e\, q_{(p)} \dot{r}^i A_i(\vec{r}_{(p)}, t)$$

$$-e\, q_{(p)} A_o(\vec{r}_{(p)}, t) \Big) - \frac{1}{4} \int d^4x\, F^{\mu\nu}(\vec{x}, t) F_{\mu\nu}(\vec{x}, t) \ , \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\vec{r}_{(p)}$, $q_{(p)}$ denote the position and charge of the $p-th$ particle respectively.

After Dirac quantization in the $A_o = 0$ gauge, one obtains the first class Hamiltonian

$$H = \sum_{p=1}^{N} \frac{1}{2m_{(p)}} \left(p_{(p)i} + e q_{(p)} A_i(\vec{r}_{(p)}, t) \right)^2$$

$$+ \int d^3x\, \frac{1}{2} \left(|\vec{E}|^2 + |\vec{B}|^2 \right) \ , \quad (2)$$

together with the Gauss (first class) constraint:

$$\varphi \equiv \partial_i E^i - \sum_p e\, q_{(p)} \delta^3(\vec{r}_{(p)} - \vec{x}) \approx 0 \ . \quad (3)$$

In these equations e is the electromagnetic coupling constant (which in $3+1$ space is dimensionless), while

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$E^i \equiv F^{io}$ and $B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk}$ denote the electric and magnetic fields. The operators $\vec{r}_{(p)}$ and $\vec{p}_{(p)}$ are canonical conjugates, likewise $\vec{A}$ and $\vec{E}$:

$$\left[r^i_{(p)}, p_{(q)j}\right] = i\delta^i_j \delta_{pq} \ , \tag{4}$$

$$\left[A_i(\vec{x}), E^j(\vec{y})\right] = i\delta^j_i \delta^3(\vec{x}-\vec{y}) \ . \tag{5}$$

The expression $A_i(\vec{r}_{(p)}, t)$ is a shorthand for

$$A_i(\vec{r}_{(p)}, t) \equiv \int d^3\vec{x}\ \delta^3(\vec{x}-\vec{r}_{(p)})A_i(\vec{x},t) \ , \tag{6}$$

where $\delta^3(\vec{x}-\vec{r}_{(p)})$ is an operator-valued distribution acting on the Hilbert space of the p-th particle:

$$\delta^3(\vec{x}-\vec{r}_{(p)Operator})\,|\vec{r}_{(p)}\rangle = \delta^3(\vec{x}-\vec{r}_{(p)})\,|\vec{r}_{(p)}\rangle \ . \tag{7}$$

The full Hilbert space of the theory may be spanned by the basis $\prod |\vec{r}_{(p)}\rangle \otimes |\vec{A}\rangle$, constructed by taking the tensorial product of the "position" eigenstates $|\vec{r}_{(p)}\rangle$ and $|\vec{A}\rangle$ associated to the particles and the field respectivelly. The Hilbert space must be restricted to the physical space, in Dirac sense, defined by $\varphi\,|\psi_{Physical}\rangle = 0$. Also, we must identify which operators are first class, i.e., gauge invariant [remember that the Gauss constraint (3) generates spatial gauge transformations, both on particle and field operators]. It is inmediate to check that the electric and magnetic fields $\vec{E}$, $\vec{B}$, together with the particle position operator $\vec{r}_{(p)}$ and the gauge covariant momentum $\vec{p}_{(p)} + e\,q_{(p)}\vec{A}(\vec{r}_{(p)}, t)$ commute with the Gauss constraint, unlike the gauge dependent operators $\vec{p}_{(p)}$ and $\vec{A}$. It is worth mentioning that every physical observable may be constructed in terms of the first class operators mentioned above [see, for instance, expression (2) for the energy of the field-particles system].

Next, let us consider the L.R. appropriate to the theory we are dealing with. A brief review of how it works in the sourceless case will help. In the pure Maxwell theory [3–6], the gauge invariante operators $\vec{E}$ and $\vec{B}$ may be realized onto loop dependent wave funtionals $\Psi(C)$ as:

$$E^i\,\Psi(C) = e\,T^i(\vec{x}, C)\,\Psi(C) \ , \tag{8}$$

$$F_{ij}\,\Psi(C) = i/e\,\Delta_{ij}(\vec{x})\,\Psi(C) \ , \tag{9}$$

where the form factor

$$T^i(\vec{x}, C) \equiv \oint dy^i\,\delta^3(\vec{x}-\vec{y}) \ , \tag{10}$$

is a distributional vector density that encodes the information of the shape of the spatial loop C. The loop derivative of Gambini-Trías $\Delta_{ij}(\vec{x})$ [3–6] is defined as:

$$\Psi(\sigma\cdot C) = \left(1 + \sigma^{ij}\Delta_{ij}(\vec{x})\right)\,\Psi(C) \ , \tag{11}$$

with σ^{ij} being the area of an infinitesimal plaquette attached at the spatial point $\vec{x}$. Thus $\Delta_{ij}(\vec{x})$ measures how the loop dependent funtion $\Psi(C)$ changes under a small deformation of its argument C. In the loop representation, the source-free Gauss law constraint ($\partial_i E^i = 0$) is automatically satisfied, since $T^i(\vec{x}, C)$ has vanishing divergence. One can thus interpret C as a closed Faraday's line of electric flux.

In the case of particles interacting with fields, one needs to enlarge the space of states. To simplify the discussion, let us begin by considering the one particle case. The interpretation of loops as Faraday's lines of electric flux, leads in a natural way to try the following picture: consider an open path $\gamma_{\vec{r}}$ starting at the particle's position $\vec{r}$ and ending at the spatial infinity [to take into account the source-free sector, this open path might be accompanied by closed contours too]. Then, consider path-dependent wave functionals $\Psi(\gamma_{\vec{r}})$, and define the action of the electric field operator as in equation (8)

$$E^i(\vec{x})\,\Psi(\gamma_{\vec{r}}) = e\,T^i(\vec{x}, \gamma_{\vec{r}})\,\Psi(\gamma_{\vec{r}}) \ . \tag{12}$$

Then, the Gauss constraint (3) states that:

$$\begin{aligned} &\left(\ e\ \partial_i T^i(\vec{x}, \gamma_{\vec{r}}) - e\,q\,\delta^3(\vec{r}-\vec{x})\ \right)\Psi(\gamma_{\vec{r}}) \\ &\quad = e\left(\delta^3(\vec{r}-\vec{x}) - q\,\delta^3(\vec{r}-\vec{x})\right)\,\Psi(\gamma_{\vec{r}}) \\ &\quad = 0, \end{aligned} \tag{13}$$

where we have dropped the $\delta^3(\infty)$ contribution arising from the end of the path. Equation (13) implies that $q = 1$. This result provides the key to complete the picture of the kinds of paths allowed. Had we taken an incoming path instead of the outgoing one, the Gauss law had been satisfied only for $q = -1$. On the other hand, if we take a "multiple" open path, i.e., n strands outgoing (incoming) from (towards) $\vec{r}$, the allowed value for q would be n ($-n$). Finally, it is easy to see that for N charges, one must take N "bundles" of open paths, one for each charge $q_{(p)}$, having as many strands as the value of the charge, and oriented according to its sign. Hence, within this formalism there is no room for fractionary charges: a Faraday line carries one unit of electric flux e, which must be emitted from or absorbed by an integral charge $q_{(p)}$. Then one has:

$$\sum_{p=1}^{N} q_{(p)}\delta^3(\vec{x}-\vec{r}_{(p)}) = \sum_s \left(\delta^3(\vec{x}-\vec{a}_s) - \delta^3(\vec{x}-\vec{b}_s)\right) \ , \tag{14}$$

with $\vec{a}_s$ and $\vec{b}_s$ labeling the starting and ending points of the s-th "strand", and the Gauss constraint (3) becomes an identity on the physical states.

It remains to study whether or not the algebra of observables admits a realization in terms of operators acting on these path-dependent (Faraday's lines dependent) functionals $\Psi(\gamma_{\vec{r}})$. Besides the electric and magnetic fields, wich are realized as in equations (8) and (9) [remember that the paths may also be comprised by closed loops, hence the loop derivative makes sense in this context too], we prescribe:

$$p_{(p)i} + e\, q_{(p)}\, A_i(\vec{r}_{(p)}, t) \rightarrow$$

$$-iD_i(\vec{r}_{(p)}) \equiv -i\left(\frac{\partial}{\partial r^i_{(p)}} - q_{(p)}\, \delta_i(\vec{r}_{(p)})\right) , \qquad (15)$$

where $\delta_i(\vec{x})$ is the "path derivative", that acts onto path-dependent functions $\Psi(\gamma_{\vec{r}_{(p)}})$ by measuring their change when an infinitesimal open path starting at $\vec{x}$ and ending at $\vec{x}+\vec{h}$ ($\vec{h} \rightarrow 0$) is appended to the list of paths comprised in $\gamma_{\vec{r}_{(p)}}$ [11]:

$$\Psi(h \cdot \gamma_{\vec{r}_{(p)}}) = \left(1 + h^i\, \delta_i(\vec{r}_{(p)})\right)\, \Psi(\gamma_{\vec{r}_{(p)}}) . \qquad (16)$$

The $\delta_i(\vec{x})$ derivative is related with the loop derivative (11) through:

$$\Delta_{ij}(\vec{x}) = \frac{\partial}{\partial x^i}\delta_j(\vec{x}) - \frac{\partial}{\partial x^j}\delta_i(\vec{x}) . \qquad (17)$$

The gauge invariant combination $D_i(\vec{r}_{(p)})$ coincides with the derivative introduced by Mandelstam several years ago [12]. It comprises the ordinary derivative, representing the momentum operator of the particle, plus $q_{(p)}$ times the "path derivative" $\delta_i(\vec{r}_{(p)})$. The "Mandelstam operator" $D_i(\vec{r}_{(p)})$ has a nice geometric interpretation within the present formulation, as we shall see. In this representation, both particles and fields are described by geometric means: particles are labelled by points $\vec{r}_{(p)}$ (as usual), and fields by open paths. Gauge invariance restricts paths to be closed, or to start (or end) at the points where particles "live". Gauge invariant operators, on the other hand, respect the geometrical properties dictated by gauge invariance: the "position" operators $\vec{r}_{(p)}$ and $\vec{E}$, are diagonal in this representation, and act by displaying the localization and shape of the geometric configurations. In turn, the magnetic field operator computes the change in the wave functional when a small "plaquette" is added, while the covariant momentum $-iD_i(\vec{r}_{(p)})$ measures the change when both the particle and its attached "bundle" of paths are infinitesimally displaced. In both cases, the involved derivative operation fulfills the geometrical requirements imposed by gauge invariance. At this point, it should be observed that a more appropriate notation for the path dependent functionals would be $\Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$, since it displays both the path and point-dependence, which are affected by the path and ordinary derivatives respectivelly.

Finally, it can be shown that the path-space operators obey the algebra arising from the canonical commutators, i.e., they constitute a representation of the quantum theory under study. For instance, one has

$$\left[-iD_i(\vec{r}_{(p)}), -iD_j(\vec{r}_{(p)})\right]\, \Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$$

$$= q_{(p)}\, \Delta_{ij}(\vec{r}_{(p)})\Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) , \qquad (18)$$

which corresponds to the relation

$$\left[p_{(p)i} + e\, q_{(p)}\, A_i(\vec{r}_{(p)}),\, p_{(p)j} + e\, q_{(p)}\, A_j(\vec{r}_{(p)})\right]$$

$$= -ie\, q_{(p)}\, F_{ij}(\vec{r}_{(p)}) . \qquad (19)$$

Summarizing, we saw that the L.R. of the Maxwell theory coupled with point charged particles is a "Faraday's lines representation" that may be set up only if electric charges are quantized, the fundamental unit of charge being the electromagnetic coupling constant e, which in this framework is the unit of electric flux carried by each Faraday's line.

III. NON-RELATIVISTIC POINT PARTICLES INTERACTING THROUGH CHERN-SIMONS FIELD

We now turn our attention to the model described by the action,

$$S = \int dt \sum_{p=1}^{N}\Big[\frac{1}{2}m_{(p)}|\dot{\vec{r}}_{(p)}|$$

$$- e\, q_{(p)}\dot{r}^i_{(p)}A_i(\vec{r}_{(p)}, t) - e\, q_{(p)}A_o(\vec{r}_{(p)}, t)\Big]$$

$$+ \frac{\kappa}{2}\int d^3x\, \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda(x)A_\mu(x) . \qquad (20)$$

This theory has been studied throughly [9], mainly due to its relationship with anyonic statistics. Our main concern will be to discuss its L.R. formulation. To this end we need the results of the Dirac quantization of this model, which may be summarized as follows [9]. The first class Hamiltonian is given by:

$$H = \sum_{p=1}^{N}\frac{1}{2m_{(p)}}\left(\vec{p}_{(p)} - e\, q_{(p)}\vec{A}(\vec{r}_{(p)}, t)\right)^2 . \qquad (21)$$

It should be recalled that the Chern-Simons term, due to its topological character, does not contribute to the energy momentum tensor. That is why the Hamiltonian in the present case looks like that of a collection of particles in an external field. Another difference with the previous case is the commutator

$$[A_i(\vec{x}), A_j(\vec{y})] = \frac{i}{\kappa}\varepsilon^{ij}\delta^2(\vec{x}-\vec{y}) , \qquad (22)$$

which, together with the commutators of the canonical operators for free particles [i.e., equation (4)] complete the non trivial part of the algebra of the quantum theory [the remaining commutators vanish identically]. The first class constraint that replaces the Gauss law of Maxwell theory, and generates time independent gauge transformations is given by

$$\kappa\vec{B}(\vec{x}) + \sum_{(p)} e\, q_{(p)}\delta^2(\vec{x} - \vec{r}_{(p)}) \approx 0\ , \tag{23}$$

where $B(\vec{x}) \equiv -\frac{1}{2}\varepsilon^{ij}F_{ij}$ is the "magnetic field". This constraint states that on the physical sector of the Hilbert space, every particle carries an amount of "magnetic" flux proportional to its electric charge, and confined to the point where the particle is.

It can be verified that the position and velocity operators, $\vec{r}_{(p)}$ and $m_{(p)}\vec{v}_{(p)} \equiv \vec{p}_{(p)} - e\, q_{(p)}\vec{A}(\vec{r}_{(p)}, t)$ are gauge invariant. Moreover, it can be seen that on the physical sector of the Hilbert space every observable of the theory may be expressed in terms of them [9]. Then, our next task is to find a suitable realization of these operators in a geometric representation. As in the theory of the previous section, we consider the space of path dependent functionals $\Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$. The action of the path and loop derivatives $\delta_i(\vec{x})$, $\Delta_{ij}(\vec{x})$, the Mandelstam derivative $D_i(\vec{r}_{(p)})$, and the form factor $T^i(\vec{x}, \gamma)$ is defined as in the former case. Then it is easy to see that the prescription:

$$A_i(\vec{x})\ \rightarrow\ \frac{i}{e}\delta_i(\vec{x}) - \frac{e}{2\kappa}\varepsilon_{ij}T^j(\vec{x}, \gamma) \tag{24}$$

realizes the commutator (22). From this result we can obtain the velocity operator as

$$\begin{aligned} m_{(p)}v_{(p)i} =& -i\left(\frac{\partial}{\partial r^i_{(p)}} + q_{(p)}\delta_i(\vec{r}_{(p)})\right) \\ &+ \frac{e^2}{2\kappa}q_{(p)}\varepsilon_{ij}T^j(\vec{r}_{(p)}, \gamma) \\ =& -iD_i(\vec{r}_{(p)}) + \frac{e^2}{2\kappa}\, q_{(p)}\,\varepsilon_{ij}T^j(\vec{r}_{(p)}, \gamma)\ , \end{aligned} \tag{25}$$

when acting on "Faraday's lines" dependent functionals $\Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$. After some calculations one can compute the following commutators in the path representation

$$\begin{aligned} \left[m_{(p)}v^i_{(p)}, m_{(q)}v^j_{(q)}\right] &= i\varepsilon^{ij}\Big(\delta_{pq}e\, q_{(q)}\, B(\vec{r}_{(q)}) \\ &\quad + \frac{e^2}{\kappa}q_{(p)}\, q_{(q)}\delta^2(\vec{r}_p - \vec{r}_q)\Big)\ , \end{aligned} \tag{26}$$

$$\left[r^i_{(p)}, m_{(q)}v^j_{(q)}\right] = i\,\delta^{ij}\delta_{pq}\ , \tag{27}$$

$$\left[r^i_{(p)}, r^j_{(q)}\right] = 0\ , \tag{28}$$

and check that they agree with what it is obtained when the same commutators are calculated directly from the canonical ones, i.e., from equations (22) and (4) [9].

Our next step will consist on studying the gauge constraint (23). Substituting equation (24) into equation (23), we find

$$\begin{aligned} &\Bigg\{\frac{i}{2e}\varepsilon^{ij}\Delta_{ij}(\vec{x}) + \frac{e}{2\kappa}\sum_s\Big(\delta^2(\vec{x} - \vec{a}_s) - \delta^2(\vec{x} - \vec{b}_s)\Big) \\ &\qquad - \frac{e}{\kappa}\sum_{p=1}^N q_{(p)}\delta^2(\vec{x} - \vec{r}_{(p)})\Bigg\}\ \Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) = 0\,. \end{aligned} \tag{29}$$

The first two terms of this expression come from the realization of the magnetic field that rises from equation (24). There is a special situation in which one knows the solution of the path-dependent differential equation (29), namely, the case when the charge is proportional to the number of strands

$$q_{(p)} = \alpha\ n_{(p)}\,. \tag{30}$$

In this case, equation (29) can be cast in the form

$$\begin{aligned} &\Bigg\{\frac{e}{2\kappa}(2\alpha - 1)\sum_s\Big(\delta^{(2)}(\vec{x} - \vec{b}_s) - \delta^{(2)}(\vec{x} - \vec{a}_s)\Big) \\ &\qquad + \frac{i}{2e}\varepsilon^{ij}\Delta_{ij}(\vec{x})\Bigg\}\ \Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) = 0\,, \end{aligned} \tag{31}$$

which we recognize as the first class constraint of the abelian Maxwell-Chern-Simons theory in an open-path representation [13]. There is a subtlety which does not spoil the similarity between the constraints of both theories: in the present study the points $\vec{r}_{(p)}$ are "ocupied" by two entities, the charged particles that may be displaced by means of $\partial/\partial r^i_{(p)}$, and the boundaries of the paths that respond to the action of the path derivative $\delta_i(\vec{r}_{(p)})$. In the Maxwell-Chern-Simons case, on the other hand, there only exist objects of the second type.

The solution of (31) is given by [13]

$$\Psi(\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) = exp\left(i\frac{e^2(2\alpha - 1)}{4\pi\kappa}\Delta\Theta(\gamma)\right)\Phi(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}), \tag{32}$$

where $\Phi(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$ is a function that depends on the path $\gamma_{\vec{r}_{(p)}}$ only through its boundary $\partial\gamma_{\vec{r}_{(p)}}$, and $\Delta\Theta(\gamma)$ is the sum of the angles subtended by the pieces of the path γ from their final points $\vec{b}_s$, minus the sum of the angles subtended by these pieces measured from their starting points $\vec{a}_s$:

$$\Delta\Theta(\gamma) \equiv \sum_s\int_\gamma dx^k\,\varepsilon^{lk}\left[\frac{(x - b_s)^l}{|\vec{x} - \vec{b}_s|^2} - \frac{(x - a_s)^l}{|\vec{x} - \vec{a}_s|^2}\right]\ . \tag{33}$$

At this point one should verify whether the gauge invariant operators of the theory preserve the form of the physical states given by equation (32). It is found that this is so, provided that $\alpha = 1$. For instance, one has for the velocity operator

$$
\begin{aligned}
& m_{(p)} v_{(p)i} \left[exp\left(i \frac{e^2}{4\pi\kappa} \Delta\Theta(\gamma) \right) \Phi(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) \right] \\
& = exp\left(i \frac{e^2}{4\pi\kappa} \Delta\Theta(\gamma) \right) \times \\
& \quad \left\{ \frac{q_{(p)} e^2}{2\pi\kappa} \sum_s \left[\frac{(r_{(p)} - b_s)^i}{|\vec{r}_{(p)} - \vec{b}_s|^2} - \frac{(r_{(p)} - a_s)^i}{|\vec{r}_{(p)} - \vec{a}_s|^2} \right] \right. \\
& \qquad \qquad \left. - i\varepsilon^{ij} D_j(\vec{r}_{(p)}) \right\} \\
& \quad \times \Phi(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) \\
& = exp\left(i \frac{e^2}{4\pi\kappa} \Delta\Theta(\gamma) \right) \Phi'(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)}) \,, \qquad (34)
\end{aligned}
$$

where Φ', in the last line, is a boundary-dependent functional, likewise Φ. Hence, we find that a consistent solution of the gauge constraint is given by equation (32), in the case where the charges of the particles coincide with their number of attached strands. As in the Maxwell-Chern-Simons case [13] there is an unitary transformation that allows us to eliminate the path dependent phase $\chi(\gamma) \equiv i \frac{e^2}{4\pi\kappa} \Delta\Theta(\gamma)$. It is given by:

$$
\Psi(\gamma, \vec{r}) \rightarrow \widetilde{\Psi}(\partial\gamma, \vec{r}) = exp\left[-\chi(\gamma)\right] \Psi(\gamma, \vec{r}) \,, \qquad (35)
$$

$$
A \,\rightarrow\, \widetilde{A} = exp\left[-\chi(\gamma)\right]\, A \, exp\left[\chi(\gamma)\right] \,, \qquad (36)
$$

with A being any gauge invariant operator of the theory. Once this transformation is performed, the path dependence of the wave functional $\widetilde{\Psi}$ is reduced to the boundary $\partial\gamma_{\vec{r}_{(p)}}$ of the path, which is just the set $\{\vec{r}_{(p)}\}$ of points occupied by the particles.

A moments thought leads one to realize that, at this point, the boundary dependence of the wave functional becomes redundant, and it suffices to employ ordinary wave functions $\Psi(\vec{r}_{(p)})$, instead of the "boundary dependent" functionals $\Psi(\partial\gamma_{\vec{r}_{(p)}}, \vec{r}_{(p)})$. At the same time, we should replace the Mandelstam derivative $D_i(\vec{r}_{(p)}) = \frac{\partial}{\partial r^i_{(p)}} + q_{(p)}\, \delta_i(\vec{r}_{(p)})$ by the ordinary "point" derivative $\frac{\partial}{\partial r^i_{(p)}}$. The Schrödinger equation of the model may then be written down as

$$
i\,\partial_t \psi(\vec{r}_{(p)}, t) = \left[\frac{1}{2} \sum_{p=1}^{N} m_{(p)} v^2_{(p)} \right] \, \psi(\vec{r}_{(p)}, t) \qquad (37)
$$

with $m_{(p)} v^i_{(p)}$ given by

$$
m_{(p)} v^i_{(p)} = p^i_{(p)} - e q_{(p)} \frac{1}{2\pi\kappa} \varepsilon^{ij} \sum_{q \neq p} e q_{(q)} \frac{(r^j_{(p)} - r^j_{(q)})}{|\vec{r}_{(p)} - \vec{r}_{(q)}|^2} \,, \qquad (38)
$$

and then we recover the well known description of the quantum mechanics of non-relativistic particles interacting through a quantized Chern-Simons field [9] that gives rise to a model of anyons. This fact should be seen as the basic justification for choosing the charge quantization scheme that we adopted in the Chern-Simons case.

IV. CONCLUSION

We have studied the L.R. quantization of point particles interacting by means of Maxwell and Chern-Simons fields. In both cases we found that the appropriate Hilbert space is made of wave functionals whose arguments are Faraday's lines emanating from or ending at the particles positions. In the Maxwell case, since the lines of force carry an amount of electric flux that must be a multiple of the coupling constant e, we find that electric charge must be quantized in order to have a consistent formulation. In the Chern-Simons case, on the other hand, the quantization of the electric charge allows to relate, in a simple form, the geometric representation of the model with the quantum mechanics of anyons as discussed in references [9,10]. We think that this feature justifies the choice of the charge quantization prescription to solve the gauge constraint (31). Hence, in the Chern-Simons case, we obtain the following picture: the paths may be "erased" by means of an unitary transformation if we prescribe that the charge is quantized.

We want to underline how gauge invariance is maintained within the geometrical framework we have presented. For instance, the covariant momentum is a generalized derivative, that translates both the charges and their associated bundles of force lines. In a similar manner, every gauge invariant operator respects the geometrical setting where the theory is represented.

It seems possible to develop a similar formulation for models of extended objects interacting through abelian p-forms. It would also be interesting to explore whether or not charge quantization is necessary for the consistence of the L.R. of the model of charged fields (instead of particles) in electromagnetic interaction.

[1] R. Gambini and J. Pullin, "Loops, knots, gauge theories and quantum gravity," *Cambridge, UK: Univ. Pr. (1996).*
[2] C. Rovelli and P. Upadhya, gr-qc/9806079.
[3] C. di Bartolo, F. Nori, R. Gambini and A. Trias, Lett. Nuovo Cim. **38**, 497 (1983).
[4] R. Gambini and A. Trias, Phys. Rev. D **27**, 2935 (1983).
[5] C. Rovelli and L. Smolin, Nucl. Phys. B **331**, 80 (1990).
[6] C. Rovelli and L. Smolin, Phys. Rev. D **52**, 5743 (1995).
[7] A. Corichi and K. V. Krasnov, quantization," hep-th/9703177.
[8] A. Corichi and K. V. Krasnov, Mod. Phys. Lett. A **13**, 1339 (1998).
[9] R. Jackiw, Annals Phys. **201**, 83 (1990).
[10] R. Banerjee and B. Chakraborty, system," Phys. Rev. D **49**, 5431 (1994).
[11] J. Camacaro, R. Gaitan and L. Leal, Mod. Phys. Lett. A **12** (1997).
[12] S. Mandelstam, Annals Phys. **19**, 25 (1962).
[13] L. Leal and O. Zapata, Phys. Rev. D **63**, 065010 (2001).
[14] L. Leal, representation," Mod. Phys. Lett. A **7**, 3013 (1992).